“A teacher of mathematics has a great opportunity. If the teacher fills the allotted time with drilling students with routine operations, the teacher kills their interest, hampers their intellectual development, and misuses the opportunity. But if the teacher challenges the curiosity of the students by setting them problems proportionate to their knowledge, and helps them solve their problems with stimulating questions, the teacher may give them a taste for, and some means of independent thinking.”

George Polya
How to Solve It. 1957

“Problem solving means engaging in a task for which the solution method is not known in advance. In order to find a solution, students must draw on their knowledge, and through this process, they will often develop new mathematical understandings. Solving problems is not only a goal of learning mathematics but also a major means of doing so. Students should have frequent opportunities to formulate, grapple with, and solve complex problems that require a significant amount of effort and should then be encouraged to reflect on their thinking.”

—Principles and Standards for School Mathematics

Being able to reason is essential to understanding mathematics. By developing ideas, exploring phenomena, justifying results, and using mathematical conjectures in all content areas and—with different expectations of sophistication—at all grade levels, students should see and expect that mathematics makes sense. Building on the considerable reasoning skills that children bring to school, teachers can help students learn what mathematical reasoning entails.”

—Principles and Standards for School Mathematics

The activities in this chapter are designed to help you improve your problem-solving, reasoning, and communication skills. Good problem solvers need to know a variety of techniques for solving problems, so the primary focus of the activities is on developing and applying a variety of problem-solving strategies: look for a pattern, make a table, use logical reasoning, make a model, and elimination.

As you complete the activities, you will learn to make conjectures based on observations and data, to verify and generalize the conjectures, and to communicate your results to others. This is the essence of mathematical inquiry. Problem solving, reasoning, and communication are the processes you will use throughout this book to explore and develop mathematical concepts.
Activity 1: When You Don’t Know What to Do

PURPOSE
Introduce a four-step approach to solving problems.

MATERIALS
Pouch: Colored Squares
Online: Half-centimeter Graph Paper

GROUPING
Work individually or in pairs.

GETTING STARTED
Problem solving has been described as “what you do when you don’t know what to do.” As you investigate the following problem, you will learn a four-step approach that may help you solve problems.

Eighteen squares can be arranged into two congruent staircases in which no squares overlap and each step up contains exactly one less square than the step below it. One way this can be done is shown at the left.

Eight squares cannot be arranged into two staircases in this way.

Non-congruent staircases
Not staircases
Step up has two less squares than the one below

What other numbers of squares can be arranged into two congruent staircases in this way?

UNDERSTAND THE PROBLEM

The first step in the four-step approach is to make sure you understand the problem. Among other things, this involves reading the problem carefully, sometimes several times, to be certain you understand what the question is. You must also identify the information needed to solve the problem and determine whether any of it is missing.

1. Describe what is meant by a staircase in this problem.
2. What does it mean for shapes to be congruent?
3. a. Arrange 30 squares into two congruent staircases.
   b. How many different ways can you do it?
4. Restate the problem in your own words.
5. How did answering these questions help you understand the problem?
MAKE A PLAN

Once you are sure you understand the problem, the next step is to *make a plan* for solving it. This often involves choosing a problem-solving strategy or strategies to use in solving the problem.

1. Describe how the problem could be modeled using graph paper or squares.
2. What strategies other than *make a model* might be useful in solving the problem?
3. Describe how you would attempt to solve the problem.

CARRY OUT THE PLAN

*Carrying out the plan* involves using your chosen strategies to attempt to solve the problem. If a particular approach doesn’t work, you may need to alter your plan.

1. Carry out your plan for solving the problem.

LOOK BACK

The final step is to *look back* over your solution not just to check your work, but to see what you have learned from solving the problem. Looking back involves checking that your answer is reasonable and that it answers the question that was asked. It also involves looking for other ways to solve the problem and looking for connections to other problems or mathematical ideas.

1. Does your solution include a way to test whether or not a given number of squares can be arranged into two congruent staircases? If not, try to find a way.
2. Try to verify your solution by solving the problem a different way.
3. a. How many different ways can 120 squares be arranged into two congruent staircases?
   b. How can you determine how many different ways a given number of squares can be arranged into congruent staircases?
4. a. How is the staircase problem related to finding the whole number factors of a number?
   b. How is it related to finding the odd and even factors?
Activity 2: Ten People in a Canoe

PURPOSE
Introduce the simplify problem-solving strategy and apply the make a table, make a model, and patterns strategies.

MATERIALS
Pouch: Five each of two different-colored squares

GROUPING
Work individually or in groups of 2 or 3.

Ten people are fishing from a canoe. The seats in the canoe are just wide enough for one person to sit on, and the center seat is empty. The five people in the front of the canoe want to change seats and fish from the back of the canoe, and the five people in the back of the canoe want to fish from the front. Because the canoe is so narrow, only one person may move at a time. A person changing seats may move to the next empty seat, or step over one other person to reach an empty seat. Any other move will capsize the canoe.

What is the minimum number of moves needed to exchange the five people in the front with the five in the back?

HINT: Sometimes, the best approach to solving a problem is to simplify it by considering easier cases of the same problem. Use squares of two different colors to represent the people in the canoe and a model like the one below to represent the seats.

Simplify the problem by solving easier cases. The solution to the problem for two people in the canoe is shown below.

Start
Move 1 (R)
Move 2 (L)
Move 3 (R)
1. Solve the problem for four people. Record the results in the table below.

<table>
<thead>
<tr>
<th>Number of People</th>
<th>2</th>
<th>4</th>
<th>6</th>
<th>8</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of Pairs</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
</tr>
<tr>
<td>Minimum Number of Moves</td>
<td>3</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Sequence of Moves</td>
<td>RLR</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

2. Complete the table.

3. Look for two patterns, one for how the moves should be made and one for the minimum number of moves. Describe the patterns you found.

4. How many people in the canoe would produce the following sequence of moves?

   R LL RRR LLLL RRRRR LLLLL RRRRRR LLLLLL
   RRRRR LLLL RRR LL R

5. If 30 people were in the canoe, how many moves would be needed for them to change places?

6. How would the results change if there was an odd number of people in the canoe?
The Legend of the Tower of Brahma

It is said that in a temple at Benares, India, the priests work continuously moving golden disks from one diamond needle to another. It seems that when the world was created, the priests of Benares were given three diamond needles and 64 golden disks. The priests were told that they were to place the disks on one of the needles in decreasing order of size and then move the whole pile to one of the other two needles, moving only one disk at a time and never placing a larger disk on top of a smaller one. According to the legend, God told the priests, “When you finish moving the pile, the world will end.”

We can simulate the priests’ problem by using coins, Cuisenaire rods, or different-sized squares cut from paper to represent the disks. Each peg can be represented by a square in a model like the one below.

Stack the objects in one of the squares in decreasing order of size. The goal is to move the stack of objects from one square to another in the fewest possible moves. There are two rules: (a) only one object may be moved at a time, and (b) a larger object may never be placed on top of a smaller one.

1. What is the minimum number of moves required to move five objects from one square to another? **HINT:** Look for two patterns, as in the previous problem.

2. Suppose the priests move one disk every second without stopping. How long will it take them to move:

   a. 10 disks?
   b. 30 disks?
   c. 50 disks?
   d. all 64 disks?
Activity 3: What’s the Number?

PURPOSE
Apply the elimination problem-solving strategy.

GROUPING
Work individually or in groups of 3 or 4.

GETTING STARTED
Use the process of elimination to solve the following number puzzles.

1. Circle the number below that is described by the following clues. Keep a record of the order in which you use the clues.
   a. The sum of the digits is 14.
   b. The number is a multiple of 5.
   c. The number is in the thousands.
   d. The number is not odd.
   e. The number is less than 2411.

   2660  2570  905
   1580  1058  1922
   1355  1455  770
   2290  2435  1770
   1832  860  1680

2. What clue or combination of clues did you use first? Why?

1. Solve the following number riddle.
   • I am a positive integer.
   • All my digits are odd.
   • I am equal to the sum of the cubes of my digits.
   • I am less than 300.

   Who am I? ________________

2. In what order did you use the clues? Why?

Rebecca has a collection of basketball cards. When she puts them in piles of two, she has one card left over. When she puts them in piles of three or four, there is also one card left over, but when she puts them in piles of five there are no cards left over.

If Rebecca has fewer than 100 basketball cards, what are the possible numbers of cards she could have?
Activity 4: Eliminate the Impossible

**PURPOSE**
Introduce the method of indirect reasoning.

**GROUPING**
Work individually or in groups of 2 or 3.

Andrea was visiting her Uncle Ralph, who has a large gumball collection. When she asked if she could have some, he said yes, but only if she could solve a problem for him. He told her that he has three jars, each covered so that no one can see the color of the gumballs. One jar is labeled red, the second green, and the third red-green. However, he said, no jar has the correct label on it. She could reach into one jar and take one gumball. Then she had to tell him the correct color of the gumballs in each jar. She reached into the jar labeled red-green and pulled out a red gumball.

1. Are there any green gumballs in that jar? Why?

2. What is the correct label for the jar labeled red-green? Explain your answer.

3. Can the jar labeled red contain red and green gumballs? Why?

4. What are the correct labels for each of the jars?

Jorge claims that he has a certain combination of U.S. coins but he cannot make change for a dollar, half dollar, quarter, dime, or nickel. Is this possible? If so, what is the greatest amount of money Jorge could have, and what coins would they be? He does not have any dollar coins.

Greatest amount: ________

Coins: __________________________
Students in the fifth grade were playing a trivia game involving states, state birds, and state flowers. They knew that in Alaska, Alabama, Oklahoma, and Minnesota, the state flowers are the camellia, forget-me-not, pink-and-white lady’s slipper, and mistletoe. The state birds are the common loon, yellowhammer, willow ptarmigan, and scissor-tailed flycatcher. No one knew which bird or flower matched which state. They called the library and received the following clues. Use the clues to complete the table below.

<table>
<thead>
<tr>
<th>State</th>
<th>Flower</th>
<th>Bird</th>
</tr>
</thead>
</table>

a. The flycatcher loves to nest in the mistletoe.
b. The forget-me-not is from the northernmost state.
c. Loons and lady’s slippers go together, but Minnesota and mistletoe do not.
d. The yellowhammer is from a southeastern state.
e. The willow ptarmigan is not from the camellia state.

Which of the clues were the key(s) to solving the puzzle? Explain your reasoning.

Each year, the Calaveras County Frog Jumping Contest is held at Angel’s Camp, California. In last year’s contest, four large bullfrogs—Flying Freddie, Sailing Susie, Jumping Joe, and Leaping Liz—captured the first four places. Each frog was decorated with a brightly colored bow before the competition began. From the following clues, determine which frog won each place and the color of its bow.

a. Joe placed next to the frog with the purple bow.
b. The frog with the yellow bow won, and the frog with the purple bow was second.
c. The colors on Freddie’s bow and Susie’s bow mix to form orange.
d. The color of the remaining bow was green.

Construct a table similar to the one above to help organize your work.
Activity 5: An Ancient Game

**PURPOSE**
Introduce the work backward problem-solving strategy.

**MATERIALS**
Other: One calculator

**GROUPING**
Work in pairs.

**GETTING STARTED**
This is a version of a game called NIM. It is a game for two players. Beginning with 17, the players alternate turns subtracting 1, 2, or 3 from the number on the calculator display. The player who makes the display read 0 is the winner.

1. Play the game several times.

2. a. In the Sample Game, what number could Player B have subtracted on his/her last turn and been sure to win the game? Explain.

   b. Find a strategy for winning the game.

3. There are many variations of this game.

   a. Try starting with 25 and subtracting 1, 2, 3, or 4. How does this change the strategy for winning the game?

   b. What is the strategy for winning the following version of NIM? Start with 47 and alternate turns subtracting 3, 5, or 7 from the number on the display. The first player to get a number less than or equal to 0 on the display is the winner.

---

**SAMPLE GAME**

<table>
<thead>
<tr>
<th>Player</th>
<th>Keys Pressed</th>
<th>Display</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>17 [−] 3 [\equiv]</td>
<td>14</td>
</tr>
<tr>
<td>B</td>
<td>[−] 2 [\equiv]</td>
<td>12</td>
</tr>
<tr>
<td>A</td>
<td>[−] 3 [\equiv]</td>
<td>9</td>
</tr>
<tr>
<td>B</td>
<td>[−] 1 [\equiv]</td>
<td>8</td>
</tr>
<tr>
<td>A</td>
<td>[−] 3 [\equiv]</td>
<td>5</td>
</tr>
<tr>
<td>B</td>
<td>[−] 3 [\equiv]</td>
<td>2</td>
</tr>
<tr>
<td>A</td>
<td>[−] 2 [\equiv]</td>
<td>0</td>
</tr>
</tbody>
</table>

Player A wins!
Activity 6: Magic Number Tricks

PURPOSE  Translate verbal phrases into algebraic expressions and reinforce the working backward problem-solving strategy.

MATERIALS  Other: Calculator

GROUPING  Work individually.

1. Dr. Wonderful, the Mathematical Magician, astounds crowds with his amazing ability to read people’s minds. Here are the directions that he gives to five people in the crowd. Follow Dr. Wonderful’s directions and complete the table below. Choose a different number for each person.

<table>
<thead>
<tr>
<th>PERSON</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Pick any number.</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3. Add 30.</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4. Divide by 3.</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5. Subtract your original number.</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

2. Dr. Wonderful tells the people to write their answers on a sheet of paper but not to reveal them to anyone else. He closes his eyes, concentrates deeply, and then claims that he knows each person’s answer. Suppose you picked 239. What do you think Dr. Wonderful would say your answer is? Why?

3. To learn why the trick works, let $n$ be the number. Write an algebraic expression for each step and record it below.

   Step 1:
   
   Step 2:
   
   Step 3:
   
   Step 4:
   
   Step 5:

   The result is ________.
People in the crowd plead with Dr. Wonderful to teach them a magic trick that they can use with their friends. He agrees and explains the following.

Example

1. “Pick a number.” (Begin with numbers less than 15.) 4
2. “Multiply by three.” 12
3. “Add seven to the product.” 19

Dr. Wonderful says, “If you tell me your final answer, I will tell you your original number.” One person says, “19.”

Dr. Wonderful closes his eyes and thinks deeply.

1. He subtracts 7. 19 − 7 = 12
2. He divides by 3 and announces the answer. 12 ÷ 3 = 4

“Try again with a harder one,” calls the crowd.

1. “Pick a number,” he says 7
2. “Add five.” 12
3. “Multiply the sum by four.” 48
4. “Subtract seven.” 41

Dr. Wonderful’s steps can be described by the algebraic equation $4(x + 5) - 7 = \text{Answer}$.

In order to determine the starting number, Dr. Wonderful works backwards to undo the equation. For each step, he does the operation that is opposite to the one he stated.

1. Begin with 41. 41
2. Add seven. 48
3. Divide by four. 12
4. Subtract five. 7

Work in groups. Have each person in the group play the part of Dr. Wonderful who will think of a set of directions for a magic trick. Dr. Wonderful will tell the members of the group to think of a number and then read the steps to be followed.

When all have determined their answers, Dr. Wonderful will “read their minds,” announcing the starting number for each person in the group.
1. One of Dr. Wonderful’s other mind-reading tricks involves birthdays. Use a calculator and follow along with the crowd as he gives the directions. Press (\(=\)) on your calculator after each step.

1. **Enter the month of your birthday.**

2. **Multiply by 5.**

3. **Add 20.**

4. **Multiply by 4.**

5. **Subtract 7.**

6. **Multiply by 5.**

7. **Add the day of your birthday.**

8. **Subtract the number of days in a non-leap year.**

2. “Oh,” “Ah,” and “Look at that” can be heard throughout the crowd. What do people see on the display of their calculator?

3. To the right of each step in Exercise 1, write an algebraic expression that correctly describes Dr. Wonderful’s direction. Study the sequence of expressions and then explain how place value helps to explain how this “magic number trick” works.

**EXTENSION**

Write some magic tricks of your own. Write the algebraic expression for each step so that you can justify your final result. Try them out on your classmates.
Activity 7: What’s the Pattern?

PURPOSE Identify and extend numerical and pictorial patterns and explore relationships between the terms of sequences and the term numbers.

GROUPING Work individually or in groups of 2 or 3.

GETTING STARTED Fill in the blanks with the numbers or pictures that complete the sequence, and briefly explain the rule you used. In some cases, an intermediate term or the last term is given so that you can check your work.

1. $\quad$, $\quad$, $\quad$, $\quad$, $\quad$, $\quad$, $\quad$

2. $\quad$, $\quad$, $\quad$, $\quad$, $\quad$, $\quad$, $\quad$

3. $\quad$, $\quad$, $\quad$, $\quad$, $\quad$, $\quad$, $\quad$

4. 3, 4, 3, 4, 5, 3, 4, 5, ___, ___, ___, ___, ___, 7

5. 2, 5, 8, 11, ___, ___, ___, ___, ___, 29

6. 53, 46, 39, 32, ___, ___, ___, 4, ___, __

7. 1, 3, 6, 10, ___, ___, ___, ___, ___, 55

8. 4, 7, 12, 19, ___, ___, ___, ___, ___, __

9. 2, 4, 8, 16, ___, ___, ___, 256, ___, __

10. 729, 243, 81, 27, ___, ___, ___, ___, ___, $\frac{1}{27}$

11. 3, 5, 8, 13, 21, ___, ___, ___, 144, ___, __

12. Explain how the sequences in Exercises 1 and 5 and in Exercises 3 and 7 are related.

13. a. Explain how the sequences in Exercises 2 and 3 are related.

   b. How can the triangles in each term of the sequence in Exercise 3 be rearranged to illustrate this relationship?
Use the given rule to determine the first eight terms of each sequence.

**Example:** Each term is the term number times 4, plus 1.

<table>
<thead>
<tr>
<th>First Term</th>
<th>Second Term</th>
<th>Third Term</th>
<th>Fourth Term</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 × 4 + 1</td>
<td>2 × 4 + 1</td>
<td>3 × 4 + 1</td>
<td>4 × 4 + 1</td>
</tr>
<tr>
<td>5</td>
<td>9</td>
<td>13</td>
<td>17</td>
</tr>
</tbody>
</table>

1. Each term is 5 times the term number, plus 2.

<p>| | | | | | | | |</p>
<table>
<thead>
<tr>
<th></th>
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<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>7</td>
<td>8</td>
</tr>
<tr>
<td></td>
<td>12</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>9</td>
</tr>
</tbody>
</table>

2. Each term is 1 less than 3 times the term number.

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<table>
<thead>
<tr>
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</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>7</td>
<td>8</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td></td>
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<td></td>
<td>9</td>
</tr>
</tbody>
</table>

Explain how this rule describes the shapes in the sequence in Exercise 1 on page 14.

3. Each term is the term number times –3, plus 47.

<p>| | | | | | | | |</p>
<table>
<thead>
<tr>
<th></th>
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</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>7</td>
<td>8</td>
</tr>
<tr>
<td></td>
<td></td>
<td>38</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>20</td>
</tr>
</tbody>
</table>

4. Each term is 2 times the square of the term number, plus 5.

<p>| | | | | | | | |</p>
<table>
<thead>
<tr>
<th></th>
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<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>7</td>
<td>8</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>9</td>
</tr>
</tbody>
</table>

5. Each term is the term number times the next term number.

<p>| | | | | | | | |</p>
<table>
<thead>
<tr>
<th></th>
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<th></th>
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</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>7</td>
<td>8</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>9</td>
</tr>
</tbody>
</table>

Explain how this rule describes the shapes in the sequence in Exercise 2 on page 14.

6. Find the difference between successive terms in the sequences in Exercises 1–3. What do you notice? Does the same thing occur in Exercises 4 and 5? Explain.

7. If you did not know the rules for the sequences in Exercises 4 and 5, how could you find the next five terms of each sequence?

8. Find the missing terms in the following sequence

2, 4, 8, 16, 30, ___ , ___ , ___ , 186, ___

9. Compare the sequence in Exercise 8 to the sequence in Exercise 9 on page 14. Explain the difference between these two sequences.
Activity 8: What’s the Rule?

PURPOSE  Develop a procedure for determining a rule that describes the general term of an arithmetic sequence.

GROUPING  Work individually or in pairs.

GETTING STARTED  Fill in each blank to discover a method for determining the rule that generates an arithmetic sequence.

<table>
<thead>
<tr>
<th>Term Number</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Term</td>
<td>4</td>
<td>11</td>
<td>18</td>
</tr>
<tr>
<td>Difference</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

What is the constant difference? ______

<table>
<thead>
<tr>
<th>Term Number</th>
<th>Constant Difference</th>
<th>What Was Done?</th>
<th>To Get</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>× 7</td>
<td>→ 7</td>
<td>= 4</td>
</tr>
<tr>
<td>2</td>
<td>× 7</td>
<td>→ _____</td>
<td>= 11</td>
</tr>
<tr>
<td>3</td>
<td>× _____</td>
<td>→ _____</td>
<td>= _____</td>
</tr>
<tr>
<td>10</td>
<td>× _____</td>
<td>→ _____</td>
<td>= _____</td>
</tr>
<tr>
<td>50</td>
<td>× _____</td>
<td>→ _____</td>
<td>= _____</td>
</tr>
</tbody>
</table>

Write a sentence, like those in Activity 7, that states a rule for generating the terms in the sequence.

Use variables to write the rule as an equation.

For each of the following sequences:
* Fill in the missing numbers.
* Find a rule that generates the terms in the sequence.
* Determine the 25th and 100th terms of the sequence.

<table>
<thead>
<tr>
<th>Rule</th>
<th>25th Term</th>
<th>100th Term</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>9, 13, 17, 21, _____, _____, _____, _____, ...</td>
<td>_____</td>
</tr>
<tr>
<td>2.</td>
<td>2, 9, 16, 23, _____, _____, _____, _____, ...</td>
<td>_____</td>
</tr>
<tr>
<td>3.</td>
<td>–3, –1, 1, 3, _____, _____, _____, _____, ...</td>
<td>_____</td>
</tr>
<tr>
<td>4.</td>
<td>98, 96, 94, 92, _____, _____, _____, _____, ...</td>
<td>_____</td>
</tr>
<tr>
<td>5.</td>
<td>77, 74, 71, 68, _____, _____, _____, _____, ...</td>
<td>_____</td>
</tr>
</tbody>
</table>
Activity 9: Fascinating Fibonacci

PURPOSE
Use a spreadsheet to explore patterns.

MATERIALS
Other: A computer with spreadsheet software

GROUPING
Work individually or with a partner.

GETTING STARTED
Leonardo of Pisano (1170–1250), who is better known by the nickname Fibonacci, was one of the most talented mathematicians of the thirteenth century. His book Liber abaci, published in 1202, contained the fascinating, although somewhat unrealistic, problem about rabbit breeding paraphrased below.

Suppose a newborn pair of rabbits, a male and a female, is put in a field surrounded on all sides by a high wall. How many pairs will there be in one year if none of the rabbits die and, beginning at age 2 months, each pair produces another new pair every month?

1. To make sure you understand the rabbit problem, complete the following table.

<table>
<thead>
<tr>
<th>Beginning of month</th>
<th>Number of newborn pairs</th>
<th>Number of 1-month-old pairs</th>
<th>Number of pairs 2 months old or older</th>
<th>Total number of pairs</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>5</td>
<td>2</td>
<td>2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The original pair is now 1 month old.
The original pair is 2 months old and produces a new pair.
The original pair produces another new pair. The pair born in month 3 is now 1 month old.
The original pair and the pair born in month 3 are both 2 months old or older and produce new pairs.

2. a. From the second month on, how is the number of 1-month-old pairs related to the number of newborn pairs the preceding month?
   b. How can the number of 1-month-old pairs and the number of pairs 2 months old or older in any month be used to find the number of pairs 2 months old or older the next month?
   c. From the second month on, how is the number of newborn pairs each month related to the number of pairs 2 months old or older?
CREATING A SPREADSHEET

Spreadsheets were originally designed with business applications in mind, but they are also excellent problem-solving tools. Follow the steps below to use a spreadsheet to create a table like the one on the preceding page and to extend it to solve the rabbit problem.

1. a. Highlight cells A1 through E1 and select Column Width from the Format menu. Set the column width to 12 characters or 1.1 inches.
   b. Enter the column headings Month, Newborn, 1 Month Old, 2 Months Old, and Total Pairs in cells A1 through E1 respectively.
   c. Enter the numbers from the first row of the table in cells A2 through E2.

2. a. Since the number of each month is 1 more than the number of the preceding month, we can use a formula to generate the number of each month. Enter the formula “=A2 + 1” in cell A3. What happens?
   b. Highlight cells A3 through A14 and select Fill Down from the Fill menu. (If your spreadsheet does not have a Fill Down command, Copy cell A3 and Paste it into cells A4 through A14.) What happens?
   c. Why must the table go to 13 months?

3. The fact that from the second month on, the number of 1-month-old pairs is equal to the number of newborn pairs the preceding month can be used to complete column C.
   a. Select cell C3 and type “=”. Click on cell B2 and press Enter. What happened?
   b. Highlight cells C3 through C14 and Fill Down.

4. For any month, the number of pairs 2 months old or older is the sum of the number of 1-month-old pairs and the number of pairs 2 months old or older the previous month. This relationship can be used to complete column D.
   a. Enter the formula “=C2 + D2” in cell D3. (Remember, instead of typing “C2” and “D2” you can simply click on the cells.)
   b. Highlight cells D3 through D14 and Fill Down.

5. The number of newborn pairs and the number of pairs 2 months old or older each month are equal. Use this fact to complete column B.

6. a. Enter the formula “=B3 + C3 + D3” in cell E3 to find the total number of pairs for month 2. (The formula can also be entered by highlighting cells B3 through E3 and clicking on the Σ (Auto Sum) in the tool bar.)
   b. Highlight cells E3 through E14 and Fill Down to complete Column E.

7. How many rabbit pairs will there be after one year?

8. The number pattern in column E is known as the **Fibonacci sequence**. The numbers in the sequence are called **Fibonacci numbers**. How are any two consecutive terms of the Fibonacci sequence used to find the next term in the sequence?
LOOKING FOR PATTERNS

The Fibonacci numbers have many interesting properties. Creating a spreadsheet can help you explore them.

1. a. Use the result from Exercise 8 on the preceding page to create a spreadsheet that contains the first 20 Fibonacci numbers in column A. Label the column Fibonacci #s.

   b. In column B, calculate the sums $1, 1 + 1, 1 + 1 + 2, 1 + 1 + 2 + 3, \ldots$ of the Fibonacci numbers. Label the column Sums.

   c. How are the sums in column B related to the Fibonacci numbers in column A?

   d. Check your conjecture in Part c by subtracting 1 from each Fibonacci number and entering the result in column C. Label the column Fib. # – 1.

2. a. In column D, calculate the square of each Fibonacci number. Label the column Squares.

   b. In column E, calculate the sums of the squares $1, 1 + 1, 1 + 1 + 4, 1 + 1 + 4 + 9, \ldots$. Label the column Sum of Squares.

   c. In column F, calculate the products $1 \times 1, 1 \times 2, 2 \times 3, 3 \times 5, 5 \times 8, \ldots$ of consecutive Fibonacci numbers. Label the column Products. How are columns E and F related?

   d. Make a conjecture about the sum of the squares of the first $n$ Fibonacci numbers.

3. In column F, calculate the quotients $1 \div 1, 2 \div 1, 3 \div 2, 5 \div 3, 8 \div 5, \ldots$ of consecutive Fibonacci numbers. Label the column Quotients. What do you notice?

The number that the ratios of the consecutive Fibonacci numbers approach is the Golden Ratio. It arises in art, music, nature, and architecture. You will encounter the Fibonacci sequence and Fibonacci numbers again in later chapters.
## Activity 10: Paper Powers

**PURPOSE**
Develop an understanding of exponents and exponential change.

**MATERIALS**
Other: Sheets of newsprint, rulers, and calculators (scientific)

**GROUPING**
Work individually or in pairs

1. Estimate the number of times you think you can fold a sheet of newsprint if you continue to fold the result in half each time. ______________

2. Now, fold the sheet in half as many times as you can. After each fold, count the number of layers of paper and record the result for the **Number of Layers** in **Standard Form** in the table. What happens to the number of layers after each fold?

3. Starting with two folds, record the **Number of Layers** in **Factored Form** and in **Exponential Form** in the table.

<table>
<thead>
<tr>
<th>Folds</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>No. Layers (Std. Form)</td>
<td>1</td>
<td>2</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>No. Layers (Fact. Form)</td>
<td></td>
<td></td>
<td>$2 \times 2$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>No. Layers (Exp. Form)</td>
<td></td>
<td></td>
<td>$2^3$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Approx. Height (cm)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>1.0</td>
</tr>
</tbody>
</table>

4. How does your estimate for the number of folds in Exercise 1 compare to the actual number of folds you were able to make?

5. Examine the pattern of entries as you go from right to left in the Exp. Form row. If the pattern continues, what will be the correct entries for 1 fold? _____ 0 folds? _____

6. If a large sheet of newsprint is folded seven times as described above, the thickness is approximately 1.0 cm. Use this number to determine other entries for **Height of the Stack** in the table.

7. Use your calculator to extend the table and determine the number of layers needed to approximate your height.

   Your height (cm) _____ No. of layers _____ Height (from table) _____

8. a. If a sheet could be folded 30 times, would the stack reach the top of the Sears Tower, _____ an orbiting satellite, _____ the moon? _____

   Y/N       Y/N       Y/N

   b. Use the pattern in the table to determine the height after 30 folds. Describe how your answer compares to the height of the Sears Tower, the distance to an orbiting satellite, and the distance from Earth to the moon.
Activity 11: The King’s Problem

**PURPOSE**
Apply problem-solving strategies to develop an understanding of exponential growth.

**MATERIALS**
Other: Rice, measuring tools, and calculators (scientific)

**GROUPING**
Work individually or in groups of 2 or 3.

**GETTING STARTED**
Legend has it that when the inventor of the game of chess explained the game to his king, the king was so delighted he asked the man what gift he would like as a reward.

“My wants are simple,” the man replied. “If you but give me one grain of rice for the first square on the playing board, two for the second, four for the third, and so on for all sixty-four squares, doubling the number of grains each time, I will be satisfied.”

1. Suppose the king agreed to the request.
   a. How many grains of rice would the inventor receive?
      **Hint:** How would the number of grains of rice on the seventh square compare to the total number of grains on the first six squares?
   b. How would the total number of grains of rice on the black squares compare to the total number of grains on the white squares?

2. a. How much would the number of grains of rice you found in Exercise 1(a) weigh?
   b. How many bushels of rice would this be? Explain how you got your answer.
   c. How large would a building need to be to hold the rice? (Make a sketch of the building and label its dimensions.)
   d. At today’s prices, what would the retail value of the rice be?

3. Consult an almanac or the Internet to answer the following.
   a. Does the United States produce enough rice in one year to satisfy the inventor’s request? Explain.
   b. Is enough rice produced in the world in one year to satisfy the inventor’s request? Explain.
   c. How long would it take to produce the needed rice?
Chapter Summary

In this chapter, you studied some of the tools and processes used to explore and develop mathematical concepts. The goal was to help you develop your own problem-solving ability.

In Activity 1, you explored a four-step approach to problem solving. One advantage of this approach is that it gives you a way to get started on solving a problem. Many of the remaining activities developed problem-solving strategies, which are often used in conjunction with the four-step approach.

Activity 2 integrated many problem-solving techniques. You learned to simulate problems that could not be experienced firsthand, to apply the patterns strategy, and to use tables as an organizer. You also learned a new problem-solving strategy, simplify the problem—begin with a simple case of the problem and work through successively more complex cases until a general method of solution is discovered. This technique will be used extensively for investigating new mathematical concepts.

The logical reasoning strategy was developed in Activities 3 and 4. Most of the activities began with a set of clues, or premises, that were accepted as true. By reasoning logically from these premises, you could conclude something about a number or situation. This process of deriving a conclusion by reasoning logically from a set of known premises is called deductive reasoning.

Usually, you were able to reason directly from the premises to the conclusion. However, in Activity 4, you had to test possible solutions by assuming they were true. If the assumption led to a contradiction of a known fact, then you knew that the solution was not correct. This method, introduced through elimination, is known as indirect reasoning—it is used extensively in mathematics.

The work backward strategy was introduced in Activity 5 and reinforced in Activity 6 where you investigated some classic number tricks. As you analyzed the number tricks, you discovered how using variables to translate the instructions into algebraic expressions could help explain the “magic.”

The study of patterns and functions is a central theme in mathematics. In this chapter, you learned various ways to analyze patterns. In Activity 7, you learned to recognize different types of patterns.
a. Patterns like the one in Exercise 4 that have a growing core:
   34, 345, 3456, 34567 ...

b. Patterns like those in Exercise 9 that grow in a predictable way:
   2, 4, 8, 16, ...

c. Patterns like those in Exercises 5 and 10, that can be extended
   by adding a constant or dividing by a constant:
   2, 5, 8, 11, ... and 729, 243, 81, 27

d. Patterns like the one in Exercise 8 where there is a pattern in the
   differences between successive terms:
   4, 7, 12, 19, ...

While studying patterns, you learned some new terminology: term,
term number, and sequence. You also learned that for many sequences
there is a rule that relates any term of the sequence to its term number.

One method for analyzing and extending numeric sequences is
to examine the differences between the successive terms of the
sequence. In Activity 8, you found that when the differences were
constant, you could use the difference to generate a rule for the
general term of the sequence.

The methods you used to analyze and extend patterns based on your
observations are examples of inductive reasoning. You discovered
the limitations of the inductive reasoning process in Activity 7. No
matter how many initial terms of a sequence you may know, there is
generally more than one way to extend it.

Logical reasoning is the cornerstone upon which mathematics is built.
New mathematics is often discovered via inductive reasoning. But
before a conjecture arrived at inductively is accepted as a fact, it must
first be verified using deductive reasoning. It is this standard of proof
that distinguishes mathematics from the other sciences.

The activities in this chapter were intended to provide only an informal
introduction to inductive and deductive reasoning. You will learn more
about these techniques later in this book and use them throughout it.

Activity 9 explored the use of technology to communicate
mathematically. In it, you investigated a classic problem posed by
Leonardo of Pisano and discovered some fascinating properties of
the Fibonacci numbers. Activities 10 and 11 used the patterns
problem-solving strategy to introduce exponents and exponential
growth.