Chapter 2

Numeration Systems and Sets

“… understanding number and operations, developing number sense, and gaining fluency in arithmetic computation form the core of mathematics education for elementary grades … As students gain understanding of numbers and how to represent them, they have a foundation for understanding relationships among numbers.”

—Principles and Standards for School Mathematics

“Sorting, Classifying, and ordering facilitate work with patterns, geometric shapes, and data. Given a package of assorted stickers, children quickly notice many differences among the items. They can sort the stickers into groups having similar traits such as color, size, or design and order them from smallest to largest.”

—Principles and Standards for School Mathematics

The activities in this chapter make connections among many of the NCTM Standards. Using the concepts associated with sets (well-defined groups of objects), identifying the properties of the elements of a set, and sorting the elements into subsets according to specific properties are all fundamental to the structure of mathematics.

The initial activities introduce the set of whole numbers and will help you develop a deeper understanding of our numeration system. The final activities will engage you in sorting and classifying objects, describing their properties (attributes), and describing their similarities and differences. Both of these activities promote communication in mathematics. Explaining your reasoning, defending your conjectures, and evaluating input from others will all promote your greater understanding of mathematical ideas.

When engaged in problem-solving situations that involve sorting, classifying, and discriminating, reflect on the connections between these processes in science and mathematics. You will find that the direct and indirect reasoning skills you develop in this chapter will be an important asset in other problem settings throughout the book.
Activity 1: Regrouping Numbers

Purpose: Develop number sense and an understanding of place-value and regrouping concepts by constructing models of numbers.

Materials: Pouch: Base Ten Blocks
Online: Place-Value Dice (hundreds, tens, and ones) and Place-Value Operations Board

Grouping: Work individually or in pairs.

Getting Started: Begin with the tens and ones dice. One student rolls the dice; the second student uses rods and cubes to construct the two-digit number represented by the dice on the place-value operations board. The number should be constructed first using the least number of blocks possible. One of the highest value blocks should then be traded for 10 of the next lower value blocks. Each number should be recorded as shown in the example. The trading should continue until only unit cubes are used. Students then switch roles.

Example:

\[
\begin{align*}
\text{3 tens + 7 ones} & : \begin{array}{c}
\text{3 tens}
\end{array} & \begin{array}{c}
\text{7 ones}
\end{array}
\\
\text{2 tens + 17 ones} & : \begin{array}{c}
\text{2 tens}
\end{array} & \begin{array}{c}
\text{17 ones}
\end{array}
\\
\text{1 ten + 27 ones} & : \begin{array}{c}
\text{1 ten}
\end{array} & \begin{array}{c}
\text{27 ones}
\end{array}
\\
\text{37 ones} & : \begin{array}{c}
\text{37 ones}
\end{array}
\end{align*}
\]

Repeat this activity with three place-value dice.
Complete the chart.

<table>
<thead>
<tr>
<th>__ hundreds</th>
<th>_ tens</th>
<th>__ ones</th>
</tr>
</thead>
<tbody>
<tr>
<td>2 hundreds</td>
<td>4 tens</td>
<td>3 ones</td>
</tr>
<tr>
<td>1 hundreds</td>
<td>_ tens</td>
<td>3 ones</td>
</tr>
<tr>
<td>1 hundreds</td>
<td>12 tens</td>
<td>__ ones</td>
</tr>
<tr>
<td>2 hundreds</td>
<td>3 tens</td>
<td>__ ones</td>
</tr>
<tr>
<td>0 hundreds</td>
<td>23 tens</td>
<td>__ ones</td>
</tr>
<tr>
<td>__ hundreds</td>
<td>12 tens</td>
<td>23 ones</td>
</tr>
</tbody>
</table>

Explain why rewriting a number in several different ways promotes the development of number sense and illustrates multiple representations of a number.

**EXTENSION** Duplicate several charts like the one shown below. Fill in some of the parts and then have your partner complete the chart.

<table>
<thead>
<tr>
<th>__ hundreds</th>
<th>_ tens</th>
<th>__ ones</th>
</tr>
</thead>
<tbody>
<tr>
<td>_ hundreds</td>
<td>_ tens</td>
<td>__ ones</td>
</tr>
<tr>
<td>_ hundreds</td>
<td>_ tens</td>
<td>__ ones</td>
</tr>
<tr>
<td>_ hundreds</td>
<td>_ tens</td>
<td>__ ones</td>
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<td>_ hundreds</td>
<td>_ tens</td>
<td>__ ones</td>
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<td>_ hundreds</td>
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<tr>
<td>_ hundreds</td>
<td>_ tens</td>
<td>__ ones</td>
</tr>
<tr>
<td>_ hundreds</td>
<td>_ tens</td>
<td>__ ones</td>
</tr>
<tr>
<td>_ hundreds</td>
<td>_ tens</td>
<td>__ ones</td>
</tr>
</tbody>
</table>

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Activity 2: Find the Missing Numbers

**PURPOSE** Reinforce number sense and place-value concepts in a problem-solving situation.

**MATERIALS**
- Pouch: Base Ten Blocks
- Online: Place-Value Operations Board

**GROUPING** Work in pairs.

**GETTING STARTED** One student reads the clues to the other. The second student uses blocks to construct a model of the missing number(s). Students alternate turns.

1. I have four base ten blocks. Some are rods and some are units. Their value is less than 25.
   Who am I? ________________

2. I have eight base ten blocks. Some are units and some are rods. Their value is an odd number between 50 and 60.
   Who am I? ________________

3. I have six base ten blocks. Some are rods and some are units. Their value is between 20 and 40.
   Who am I? ________________

4. I have six base ten blocks. Some are flats, some are rods, and some are units. I am a palindrome.
   Who am I? ________________

5. I have two base ten blocks.
   Who am I? ________________

6. I have three base ten blocks. None of them are flats.
   Who am I? ________________

7. I have four base ten blocks. None of them are flats.
   Who am I? ________________

8. I have four base ten blocks. Only one of them is a flat.
   Who am I? ________________

**EXTENSION** Create additional clue cards for your partner to solve.
Activity 3: A Visit to Fouria

**PURPOSE** Use a game to reinforce place-value concepts, to introduce the base-four numeration system, and to develop understanding of the regrouping process.

**MATERIALS** Other: Blue, red, and white chips (10 of each color) and a die

**GROUPING** Work in pairs.

**GETTING STARTED** While on an Intergalactic Numismatics Tour you encounter a meteor shower and are forced to make an unscheduled stop on the planet Fouria. The monetary system used on Fouria consists of three coins: a white coin (worth $1 in U.S. money), a red coin, and a blue coin. The red coin is equivalent in value to four white coins, and the blue coin is equivalent to four red coins.

1. Unlike its sister planet, Ufouria, Fouria turns out to be a rather dull place to visit. To help pass the time in the waiting area, you and a fellow passenger play a coin trading game. The rules of the game are:
   - Players alternate turns.
   - On each turn, a player rolls the die and places that number of white Fourian coins in the white column on his or her Coin Trading Game Sheet.
   - Whenever possible, a player must trade four white coins for one red coin and/or four red coins for one blue coin.
   - Coins must be placed in the appropriately labeled column. No more than three coins of one color may be in any column at the end of a turn.
   - The first player to get two blue coins is the winner.

   Make a Coin Trading Game Sheet, and play the game with a partner. At the end of each turn, record the number of each color coin on your game sheet in a table like the one at the right.

| Turn | Number Rolled | Result
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
<td></td>
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<tr>
<td>5</td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td></td>
<td></td>
</tr>
<tr>
<td>9</td>
<td></td>
<td></td>
</tr>
<tr>
<td>10</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
THE COIN TRADING GAME REVISITED

1. A third passenger has been watching you play. She suggests it is more challenging to start the game with three blue coins and to remove the number of white coins equal to the number rolled on each turn. The first player to remove all the coins from his or her playing board is the winner. Your playing partner is confused. "How can you remove white coins when there aren’t any on the board?" he asks. Explain how this can be done.

2. Play this new version of the game with a partner. Each player should record the result of each move in a table like the one on the preceding page.

1. You become bored with the games and go to the spaceport newsstand to buy something to read. Glancing at the cover of a Fourian magazine you notice that the price is given as 123four. At the checkout stand, the clerk explains that this means one blue coin, two red coins, and three white coins. How many of each color coin does each of the following prices represent?
   a. 231four ______________________ b. 102four ______________________
   c. 13four ______________________ d. 20four ______________________

2. How would Fourians write each of the following prices?
   a. 1 blue, 2 red, 1 white __________ b. 2 red, 3 white ________________
   c. 2 blue ______________________ d. 2 blue, 3 white _________________

3. Back in the waiting area, you begin leafing through a magazine that you purchased. You note that the first page is numbered 1four, but when you get to the fourth page, you are surprised to find that it is numbered 10four. Fill in the blanks below to show how the remaining pages of the magazine would be numbered.

   1four _____ _____ 10four _____ _____ _____ _____ _____ _____
   _____ _____ _____ 33four _____ _____ _____ _____ _____ _____
   _____ _____ _____ _____ 333four _____ _____ _____ _____ _____
1. After reading for a while, you decide to have dinner. The price of your meal was 121\text{four}. When you go to the cashier to pay for your meal, you realize that you don’t have any Fourian money with you. “No problem,” the cashier explains. “You may pay with dollars.” What is the cost of your meal in dollars?

2. a. On your way back to the waiting area, you stop in the newsstand to buy a souvenir. It costs 1312\text{four}. How many dollars is this?
   
b. You give the cashier two $100 bills. How much Fourian money should you get back in change?

1. Back in the waiting area, you find a mathematics book left behind by a Fourian student. Flipping through the book, you come across the examples shown below. Explain what the small 1 in the second example means and how it was obtained.

\[
\begin{array}{c}
231\text{four} \\
+102\text{four} \\
\hline
333\text{four}
\end{array}
\quad
\begin{array}{c}
213\text{four} \\
+33\text{four} \\
\hline
122\text{four}
\end{array}
\]

2. Use what you learned in the examples in Exercise 1 to find the following sums.
   
a. 121\text{four} + 211\text{four} \\
b. 123\text{four} + 221\text{four}

3. A few pages later, you find the following examples. Explain what is being done in steps A, B, and C.

\[
\begin{array}{l}
333\text{four} \\
-121\text{four} \\
\hline
212\text{four}
\end{array}
\quad
\begin{array}{l}
21\text{four} \\
-12\text{four} \\
\hline
13\text{four}
\end{array}
\]

4. Use what you learned in the examples in Exercise 3 to find the following differences.
   
a. 323\text{four} − 211\text{four} \\
b. 221\text{four} − 132\text{four}

EXTENSIONS

1. Explain how the place-value system used to write Fourian numerals is related to the place-value system used to write base-ten numerals.

2. Explain how the regrouping used in Fourian addition and subtraction is related to the regrouping used in traditional (base-ten) addition and subtraction.
Activity 4: What’s in the Loop?

PURPOSE
Use the elimination and logical reasoning problem-solving strategies to develop the concept of a set, to determine the attribute defining a set, and to explore the concepts of complement, equivalent sets, equal sets, and cardinality.

MATERIALS
Pouch: Attribute Pieces
Online: Label Cards for Attribute Pieces
Other: Large loop of string

GROUPING
Work in pairs or in teams of two students each.

1. Place the loop between the players. Put the RED label card face up on the loop. Take turns placing pieces in the appropriate region, either inside the loop or outside it.

2. The pieces that are in the loop are the set of red attribute pieces. A set is any collection of objects. Each attribute piece that is in the loop is an element of the set of red attribute pieces. The universal set is the set that contains all the elements being considered in a given situation, in this case, the set of all attribute pieces.
   a. Use abbreviations to list the elements in the set of red attribute pieces. For example, use SRT for the small red triangle.
   b. The number of elements in a set is the cardinality of the set. What is the cardinality of the set of red attribute pieces?
   c. What is the cardinality of the universal set?

3. a. The complement of a set \( A \) is the set of all elements of the universal set that are not in set \( A \). List the elements in the complement of the set of red attribute pieces.
   b. Where are these pieces in relation to the loop?

4. a. Two sets that contain the same number of elements are equivalent. Name a set of attribute pieces that is equivalent to the set of red pieces.
   b. Without actually counting the number of pieces in the two sets, how could you show that the sets are equivalent?
   c. Two sets are equal if and only if they contain exactly the same elements. Name a set of attribute pieces that is equal to the set of small attribute pieces.

5. Place the loop between the players. Shuffle the label cards and place them face down on the table. Player A picks a card from the pile, looks at it, and places it face down next to the loop without showing it to the other player. Player B chooses an attribute piece and places it in the loop. Player A then tells whether the placement is correct or not based on the label card. Play continues until Player B can correctly name the set by identifying the exact attribute on the label card. Players then switch roles.
Activity 5: Loop de Loops

**PURPOSE**
Use indirect reasoning to determine the attribute defining a set, and to explore the concepts of union, intersection, empty set, and subset.

**MATERIALS**
Pouch: Attribute Pieces
Online: Label Cards for Attribute Pieces
Other: Two large loops of string

**GROUPING**
Work in pairs or in teams of two students each.

1. **a.** Place the loops between the players as shown in the diagram below. Put the LARGE label card face up on one loop and the RED label card face up on the other. Take turns placing pieces in the appropriate loop or outside the loops.

2. **a.** Complete the following sentence describing the attribute pieces that are in the shaded region.
   The shaded region is the set of all attribute pieces that are ____________________.

   **b.** The set in Part a is the union of the set of red pieces and the set of large pieces. How many pieces are in the union of the two sets, that is, how many pieces are either RED or LARGE?

3. **a.** Complete the following sentence describing the attribute pieces that are in region B.
   Region B is the set of all attribute pieces that are ____________________.

   **b.** The set in Part a is the intersection of the set of red pieces and the set of large pieces. How many pieces are in the intersection of the two sets, that is, how many pieces are both RED and LARGE?

4. The empty set is a set that contains no elements. What labels could be used for the loops so that the intersection of the two loops is empty?
5. Set $A$ is a subset of set $B$ if and only if every element of $A$ is also an element of $B$. What labels could be used for the two loops so that one loop would be a subset of the other?

6. a. How many pieces are LARGE but not RED?
   b. What region contains the pieces that are LARGE but not RED?

7. A few of the attribute pieces have been placed in the loops in the diagram below. What are the labels for the loops? How do you know?

8. Place the loops between the players. Overlap them as shown below. Shuffle the label cards and place them face down on the table. Player A picks two label cards and, without showing them to the other player, places them face down, one on each loop as shown.

   Player B chooses an attribute piece and places it in one of the four regions. Player A then indicates whether the placement is correct or not according to the labels on the cards that have been placed on the loops. If the placement is incorrect, the first student may try another region, or put the piece back in the pile and try another attribute piece. Play continues until Player B can correctly identify both label cards. Players then switch roles.

9. Repeat Exercise 8 using three loops and three label cards.
Chapter Summary

Activities 1 and 2 in this chapter introduced the set of whole numbers and developed an understanding of place-value and regrouping concepts in our base-ten numeration system. Activity 3 reinforced this understanding through the investigation of a base-four numeration system.

Sets and the operations on them were also introduced in this chapter. The fields of mathematics, such as arithmetic, geometry, and algebra, are characterized by the study of particular sets of objects. This study entails an analysis of the elements of the set, their attributes, and what makes the set uniquely different from other sets.

In the activities, you sorted the elements of a set, the universal set, according to certain attributes. In Activity 4, when you made a choice to place a particular attribute piece in the loop, you guessed (made an assumption) that a certain label belonged on that loop. Given a YES answer, you gained information about the correct label. Given a NO answer, you gained information about which label was NOT correct, thus eliminating some labels, and reducing the number of pieces to try in order to complete the problem. Activity 5 extended this use of the elimination problem-solving strategy and the use of indirect reasoning to determine the proper label for the sets.

As you completed the activities, you encountered the set concepts of union, intersection, subset, complement, equal sets, equivalent sets, and the empty set. The word OR described the union of sets, AND, the intersection of sets, and NOT, the complement. The following diagrams illustrate these concepts.